

Stability of Synchronized Motion in Complex Networks

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Preface

These lectures are based on material which was presented in the Summer school at University of São Paulo, and in the winter school at Federal university of ABC. The aim of this series is to introduce graduate students with a little background in the field to dynamical systems and network theory.

Our goal is to give a succinct and self-contained description of the synchronized motion on networks of mutually coupled oscillators. We assume that the reader has basic knowledge on linear algebra and theory of differential equations.

Usually, the stability criterion for the stability of synchronized motion is obtained in terms of Lyapunov exponents. We avoid treating the general case for it would only bring further technicalities. We consider the fully diffusive case which is amenable to treatment in terms of uniform contractions. This approach provides an interesting application of the stability theory and exposes the reader to a variety of concepts of applied mathematics, in particular, the theory of matrices and differential equations. More important, the approach provides a beautiful and rigorous, yet clear and concise, way to the important results.

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Chapter 1

Introduction

The art of doing mathematics consists in finding that special case which contains all the germs of generality.
– David Hilbert

Real world complex systems can be viewed and modeled as networks of interacting elements [1, 2, 3]. Examples range from geology [4] and ecosystems [5] to mathematical biology [6] and neuroscience [7] as well as physics of neutrinos [8] and superconductors [9]. Here we distinguish the structure of the network, the nature of the interaction, and the (isolated) dynamical behavior of individual elements.

During the last fifty years, empirical studies of real complex systems have led to a deep understanding of the structure of networks, interaction properties, and isolated dynamics of individual elements, but a general comprehension of the resulting network dynamics remains largely elusive.

Among the large variety of dynamical phenomena observed in complex networks, collective behavior is ubiquitous in real world networks and has proven to be essential to the functionality of such networks [10, 11, 12, 13]. Synchronization is one of the most pervasive form collective behavior in complex systems of interacting components [14, 15, 16, 17, 18]. Along the riverbanks in some South Asian forests whole swarms of fireflies will light up simultaneously in a spectacular synchronous flashing. Human hearts beat rhythmically because thousands of cells synchronize their activity [14], while thousands of neurons in the visual cortex synchronize their activity in response to specific stimuli [19]. Synchronization is rooted in human life, from the metabolic processes in our cells to the highest cognitive tasks [20, 21].

Synchronization emerges from the collaboration and competition of many elements and has important consequences to all elements and network functioning. Synchronization is a multi-disciplinary discipline with broad range applications. Currently, the field experiences a vertiginous growth and significant progress has already been made on various fronts.

Strikingly, in most realistic networked systems where synchronization is relevant, strong synchronization may also be related to pathological activities such as epileptic seizures [22] and Parkinson's disease [23] in neural networks, to extinction in ecology [24], and social catastrophes in epidemic outbreaks [25]. Of particular interest is how synchronization depends on various structural parameters such as degree distribution and spectral properties of the graph.

In the mid-nineties Pecora and Carroll [26] have put forward a paradigmatic model of diffusively coupled identical oscillators on complex networks. They have shown that complex networks of identical nonlinear dynamical systems can globally synchronize despite exhibiting complicated dynamics at the level of individual elements.

The analysis of synchronization in complex networks has benefited from advances in the understanding of the structure of complex networks [27, 28]. Barahona and Pecora [29] have shown that well-connected networks – with so-called small-world structure – are easier to globally synchronize than regular networks. Motter and collaborators [30] have shown that heterogeneity in the network structure hinders global synchronization. These results form only the beginning of a proper understanding of the connections between network structure and the stability of global synchronization.

The approach put forward by Pecora and Carroll, which characterized the stability of global synchronization, is based on elements of the theory of Lyapunov exponents [31]. The characterization of stability via theory of Lyapunov exponents has many additional subtleties, in particular, when it comes to the persistence of stability under perturbations. Positive solution to the persistence problem requires the analysis of the so called regularity condition, which is tricky and difficult to establish.

We consider the fully diffusive case – the coupling between oscillators depends only on their state difference. This model is amenable to full analytical treatment, and the stability analysis of the global synchronization is split into contributions coming solely from the dynamics and from the network structure. The stability conditions in this case depend only on general properties of the oscillators and can be obtained analytically if one possesses knowledge of global properties of the dynamics such as the boundedness of the trajectories. We establish the persistence under nonlinear perturbations and linear perturbations. Many conclusions guide us toward the ultimate goal of understanding more general collective behavior

Chapter 2

Graphs : Basic Definitions

Can the existence of a mathematical entity be proved without defining it?

– Jacques Hadamard

2.1 Adjacency and Laplacian Matrices

A network is a graph G comprising a set of N nodes (or vertices) connected by a set of M links (or edges). Graphs are the mathematical structures used to model pairwise relations between objects. We shall often refer to the network topology, which is the layout pattern of interconnections of the various elements. Topology can be considered as a virtual shape or structure of a network.

The networks we consider here are *simple* and *undirected*. A network is called *simple* if the nodes do not have self connections, and *undirected* if there is no distinction between the two vertices associated with each edge. A *path* in a graph is a sequence of connected (non repeated) nodes. From each of node of a path there is a link to the next node in the sequence. The length of a path is the number of links in the path. See further details in Ref. [32].

For example, lets consider the network in Fig. 2.1a). Between to the nodes 2 and 4 we have three paths $\{2, 1, 3, 4\}$, $\{2, 5, 3, 4\}$ and $\{2, 3, 4\}$. The first two have length 3, and the last has length 2. Therefore, the path $\{2, 3, 4\}$ is the shortest path between the node 2 and 4.

The network *diameter* d is the greatest length of the shortest path between any pair of vertices. To find the diameter of a graph, first find the shortest path between each pair of vertices. The greatest length of any of these paths is the diameter of the graph. If we have an isolated node, that is, a node without any connections, then we say that the diameter is infinite. A network of finite diameter is called *connected*.

A connected component of an undirected graph is a subgraph with finite diameter. The graph is called *directed* if it is not undirected. If the graph is directed then there are two connected nodes say, u and v , such that u reachable from v , but v is not reachable from u . See Fig. 2.1 for an illustration.

The network may be described in terms of its *adjacency matrix* A , which encodes the topological information, and is defined as

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise .} \end{cases}$$

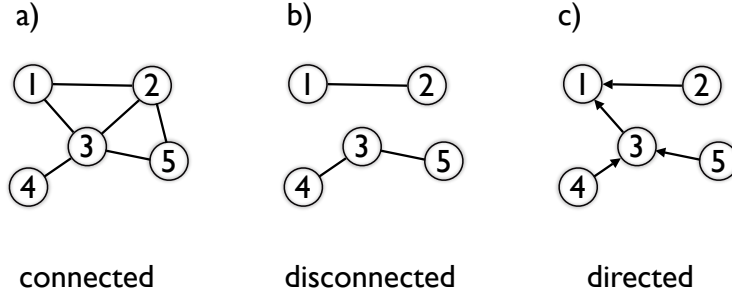


Fig. 2.1 Examples of undirected a) and b) and directed c) graphs. The diameter of graph a) is $d = 2$ hence, the graph is connected. The graph b) is disconnected, there is no path connecting the nodes 1 and 2 to the remaining nodes, the diameter is $d = \infty$. However, the graph has two connected components, the upper (1,2) with diameter $d = 1$, and the lower nodes (3,4,5) with diameter $d = 2$. Graph c) is directed, the arrow tells the direction of the connection, so node 1 is reachable from node 2, but not the other way around.

An undirected graph has a symmetric adjacency matrix. The *degree* k_i of the i th node is the number of connections it receives, clearly

$$k_i = \sum_j A_{ij}.$$

Another important matrix associated with the network is the combinatorial Laplacian matrix \mathbf{L} , defined as

$$L_{ij} = \begin{cases} k_i & \text{if } i = j \\ -1 & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} . \end{cases}$$

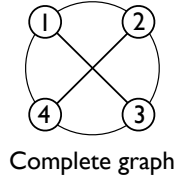
The Laplacian \mathbf{L} is closely related to the adjacency matrix \mathbf{A} . In a compact form it reads

$$\mathbf{L} = \mathbf{D} - \mathbf{A},$$

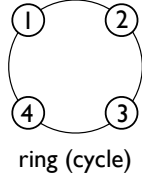
where $\mathbf{D} = \text{diag}(k_1, \dots, k_n)$ is the matrix of degrees. We depict in Fig. 2.2 distinct networks of size 4 and their adjacency and laplacian matrices.

2.2 Spectral Properties of the Laplacian

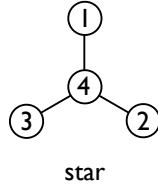
The eigenvalues and eigenvectors of \mathbf{A} and \mathbf{L} tell us a lot about the network structure. The eigenvalues of \mathbf{L} for instance are related to how well connected is the graph and how fast a random walk on the graph could spread. In particular, the smallest nonzero eigenvalue of \mathbf{L} will determine the synchronization properties of



$$A_c = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad L_c = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$



$$A_r = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad L_r = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$



$$A_s = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad L_s = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Fig. 2.2 Networks containing four nodes. Their adjacency and laplacian matrices are represented by A and L . Further details can be found in Table 2.2.

the network. Since the graph is undirected the matrix L is symmetric its eigenvalues are real, and L has a complete set of orthonormal eigenvectors 9. The next result characterizes important properties of the Laplacian

Theorem 1 *Let G be an undirected network and L its associated Laplacian. Then:*

- L has only real eigenvalues,
- 0 is an eigenvalue and a corresponding eigenvector is $\mathbf{1} = (1, 1, \dots, 1)^*$, where $*$ stands for the transpose.
- L is positive semidefinite, its eigenvalues enumerated in increasing order and repeated according to their multiplicity satisfy

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

- The multiplicity of 0 as an eigenvalue of L equals the number of connect components of G .

Proof : The statement a) follows from the fact that L is symmetric $L = L^*$, see Ap. A Theorem 9. To prove b) consider the $\mathbf{1} = (1, 1, \dots, 1)^*$ and note that

$$(\mathbf{L} \mathbf{1})_i = \sum_j L_{ij} = k_i - \sum_j A_{ij} = 0 \quad (2.1)$$

Item c) follows from the Gershgorin theorem, see Ap. A Theorem 8. The nontrivial conclusion d) is one of the main properties of the spectrum. To prove the statement d) we first note that if the graph G has r connected components G_1, \dots, G_r , then is possible to represent L such that it splits into blocks L_1, \dots, L_r .

Let m denote the multiplicity of 0. Then Each L_i has an eigenvector z_i with 0 as an eigenvalue. Note that $z_i = (z_i^1, \dots, z_i^n)$ can be defined as z_i^j is equal to 1 if j belongs to the component i and zero otherwise, hence $m \geq r$. It remains to show that any eigenvector g associated with 0 is also constant. Assume that g is a non constant eigenvector associated with 0, and let $g_\ell > 0$ be the largest entry of g . Then

$$\begin{aligned} (Lg)_\ell &= \sum_j L_{\ell j} g_j \\ &= \sum_j (k_\ell \delta_{\ell j} - A_{\ell j}) g_j, \end{aligned}$$

since g is associated with the 0 eigenvalue we have

$$g_\ell = \frac{\sum_j A_{\ell j} g_j}{k_\ell}.$$

This means that the value of the component g_ℓ is equal to the average of the values assigned to its neighbors. Hence g must be constant, which completes the proof. \square

Therefore, λ_2 is bounded away from zero whenever the network is connected. The smallest non-zero eigenvalue is known as algebraic connectivity, and it is often called the Fiedler value. The spectrum of the Laplacian is also related to some other topological invariants. One of the most interesting connections is its relation to the diameter, size and degrees.

Theorem 2 *Let G be a simple network of size n and L its associated Laplacian. Then:*

1. [33] $\lambda_2 \geq \frac{4}{nd}$
2. [34] $\lambda_2 \leq \frac{n}{n-1} k_1$

We will not present the proof of the Theorem here, however, they can be found in references we provide in the theorem. We suggest the reader to see further bounds on the spectrum of the Laplacian in Ref. [35]. Also Ref. [36] presents many applications of the Laplacian eigenvalues to diverse problems. One of the main goals in spectral graph theory is the obtain better bounds by having access to further information on the graphs.

For a fixed network size, the magnitude of λ_2 reflects how well connected is graph. Although the bounds given by Theorem 2 are general they can be tight for certain graphs. For the ring the low bound on λ_2 is tight. This implies that as the size increase – consequently also its diameter – λ_2 converges to zero, and the network becomes effectively disconnected. In sharp contrast we find the star network. In this

case, the upper bound in *i*) is tight. The star diameter equals to two, regardless the size and $\lambda_2 = 1$. See the table for the precise values.

Table 2.1 Network of n nodes. Examples of such networks are depicted in Fig. 2.2

Network	λ_2	k_n	k_1	D
Complete	n	$n-1$	$n-1$	1
ring	$2 - 2\cos\left(\frac{2\pi}{n}\right)$	2	2	$\frac{(n+1)}{2}$ if n is odd $n/2$ if n is even
Star	1	$n-1$	1	2

The networks we encounter in real applications have a wilder connection structure. Typical examples are cortical networks, the Internet, power grids and metabolic networks [1]. These networks don't have a regular structure of connections such as the ones presented in Fig. 2.2. We say that the network is *complex* if it does not possess a regular connectivity structure.

One of the goals is the understand the relation between the topological organization of the network and its relation functioning such as its collective motion. In Fig. 2.3, we depict two networks used to model real networks, namely the Barabasi-Albert and the Erdos-Renyi Networks.

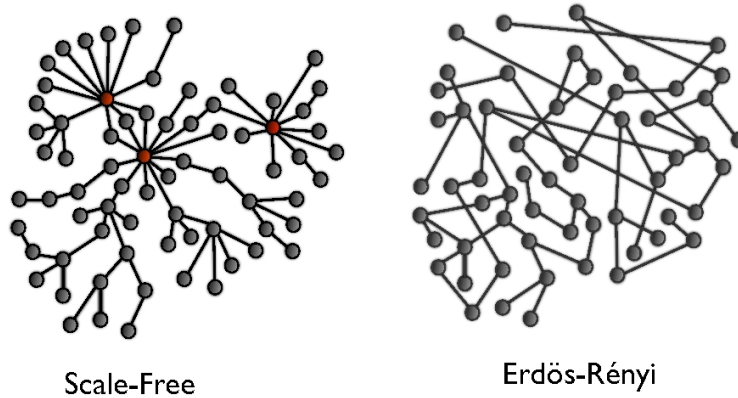


Fig. 2.3 Some examples of complex networks.

The Erdős-Rényi network is generated by setting an edge between each pair of nodes with equal probability p , independently of the other edges. If $p \gg \ln n/n$, then a the network is almost surely connected, that is, as N tends to infinity, the

probability that a graph on n vertices is connected tends to 1. The degree is pretty homogeneous, almost surely every node has the same expected degree [27].

The Barabasi-Albert network possesses a great deal of heterogeneity in the node's degree, while most nodes have only a few connections, some nodes, termed hubs, have many connections. These networks do not arise by chance alone. The network is generated by means of the cumulative advantage principle – the rich gets richer. According to this process, a node with many links will have a higher probability to establish new connections than a regular node. The number of nodes of degree k is proportional to $k^{-\beta}$. These networks are called scale-free networks [1]. Many graphs arising in various real world networks display similar structure as the Barabasi-Albert network [2, 3].

Chapter 3

Nonlinear Dynamics

How can intuition deceive us at this point ?
– Henri Poincaré

Let D be an open simply connected subset of \mathbb{R}^m , $m \geq 1$, and let $f \in C^r(D, \mathbb{R}^m)$ for some $r \geq 2$. We assume that the differential equation

$$\frac{dx}{dt} = f(x) \quad (3.1)$$

models the dynamics of a given system of interest. Now since f is differentiable the Picard-Lindelöf Theorem guarantees the existence of local solutions, see Ap. B Theorem 16. We wish to guarantee that the solutions also exist globally. This requires further hypothesis on the behavior of the vector field. We are interested in systems that dissipate the volumes of \mathbb{R}^m – called dissipative systems.

3.1 Dissipative Systems

We say that set $\Omega \subset \mathbb{R}^m$ under the dynamics of Eq. (3.1) is positively invariant if the trajectories starting at the set never leave it in the future, that is, if $x(t_0) \in \Omega$ then $x(t) \in \Omega$ for all $t \geq t_0$. Intuitively, it means that once the trajectory enters Ω it never leaves it again. The system is called dissipative if the solutions enter a positively invariant set $\Omega \subset D$ in finite time. Ω is called absorbing domain of the system. The existence of an absorbing domain guarantees that the solutions are bounded, hence, the extension result in Ap. B Theorem 17 assures the global existence of the solutions.

The question is then how to obtain the absorbing domains. Note that whenever f is nonlinear finding the solutions of Eq. 3.1 can be a rather intricate problem. And usually we won't be able to do it analytically. So we need new machinery to address the problem on absorbing domains. A method by Lyapunov allows us to obtain such domains without finding the trajectories. The technique infers the existence of the absorbing domains in relation to some properties of a scalar function – the Lyapunov function.

We will study notions relative to connected nonempty subsets Ω of \mathbb{R}^m . A function $V : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be positive definite with respect to the set B if $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \Omega$. It is radially unbounded if

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} V(\mathbf{x}) = \infty.$$

Note that this condition guarantees that all level sets of V are bounded. This fact plays a central role in the analysis. We also define $V' : \mathbb{R}^m \rightarrow \mathbb{R}$ as

$$V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

where \cdot denotes the Euclidean inner product. This definition agrees with the time derivative along the trajectories. That is, if $\mathbf{x}(t)$ is a solution of Eq. (3.1), then by the chain rule we have

$$\frac{dV(\mathbf{x}(t))}{dt} = V'(\mathbf{x}(t)).$$

the main result is then the following

Theorem 3 (Lyapunov) *Let $V : \mathbb{R}^m \rightarrow \mathbb{R}$ be radially unbounded and positive definite with respect to the set $\Omega \subset D$. Assume that*

$$V'(\mathbf{x}) < 0 \text{ for all } \mathbf{x} \in \mathbb{R}^m \setminus \Omega$$

Then all trajectories of Eq. (3.1) eventually enter the set Ω , in other words, the system is dissipative.

Proof: Note that for any trajectory $\mathbf{x}(t)$ in virtue of the fundamental theorem of the calculus

$$V(\mathbf{x}(t)) - V(\mathbf{x}(s)) = \int_s^t V'(\mathbf{x}(u)) du < 0.$$

So $V(\mathbf{x}(t)) < V(\mathbf{x}(s))$ for any $t > s$, and V is decreasing along solutions and is radially unbounded, the level sets

$$S_a = \{\mathbf{x} \in \mathbb{R}^m : V(\mathbf{x}) \leq a\}$$

are positively invariant. Hence, the solutions are bounded, and will lie in smaller level sets as time increase until the trajectory enters Ω . It remains to show that once the solutions lie in Ω they don't leave it.

Suppose $\mathbf{x}(t)$ leaves Ω at t_0 and let $b = V(\mathbf{x}(t_0))$. The level set S_b is closed, and there is a ball $B_r(\mathbf{x}(t_0))$ such that $\mathbf{x}(t_0 + \varepsilon) \in B_r(\mathbf{x}(t_0)) \setminus S_b$ for some small ε . Hence, $V(\mathbf{x}(t_0 + \varepsilon)) > V(\mathbf{x}(t_0))$ contradicting the fact that V is decreasing along solutions. \square

There are also converse Lyapunov theorems [37]. Typically if the system is dissipative (and have nice properties) then there exists a Lyapunov function. Although the above theorem is very useful, since we don't need knowledge of the trajectories, the drawback is the function V itself. There is no recipe to obtain a function V fulfilling all these properties. One could always try to guess the function, or go for a

general form such as choosing a quadratic function V . We assume that the Lyapunov function is given.

Assumption 1 *There exists a symmetric positive matrix \mathbf{Q} such that*

$$V(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{a})^* \mathbf{Q}(\mathbf{x} - \mathbf{a}).$$

where $\mathbf{a} \in \mathbb{R}^m$. Consider the set $\Omega := \{\mathbf{x} \in \mathbb{R}^m \mid (\mathbf{x} - \mathbf{a})^\dagger \mathbf{Q}(\mathbf{x} - \mathbf{a}) \leq \rho^2\}$, then

$$V'(\mathbf{x}) < 0, \forall \mathbf{x} \in \mathbb{R}^m \setminus \Omega.$$

Under Assumption 1, Theorem 3 guarantees that Ω is positively invariant and that the trajectories of Eq. (3.1) eventually enter it. So, Ω is the absorbing domain of the problem. The solutions are, therefore, globally defined.

3.2 Chaotic Systems

Since the system Eq. (3.1) is dissipative the solutions accumulates in a neighborhood of a bounded set $\Lambda \subset \Omega$. The set Λ is called attractor. We focus on the situation where Λ is a chaotic attractor. Now, the definition of a chaotic attractor is rather intricate – there is even a general definition, the important properties for us are that solutions on the attractor are aperiodic, i.e., there is no $\tau \geq 0$ such that $\mathbf{x}(t) = \mathbf{x}(t + \tau)$, and the solutions exhibits sensitive dependence on initial conditions. Sensitive dependence on initial conditions means that nearby trajectories separate exponentially fast.

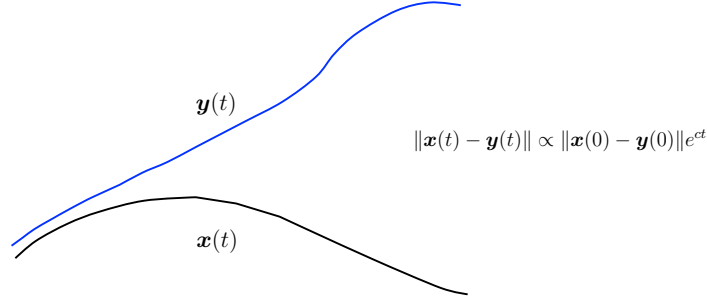


Fig. 3.1 If $\mathbf{x}(0) \neq \mathbf{y}(0)$ are both in a neighborhood of the attractor, then for small times $\|\mathbf{x}(t) - \mathbf{y}(t)\| \propto \|\mathbf{x}(0) - \mathbf{y}(0)\|e^{ct}$, for some $b > 0$.

If the system is chaotic no matter how close two solutions start, they move apart when they are close to the attractor. Hence, arbitrarily small modifications of ini-

tial conditions typically lead to quite different states for large times. This sensitive dependence on initial conditions is one of the main features of a chaotic system. Exponential divergency cannot go on for ever, since the attractor is bounded, it is possible to show that the trajectories will come close together in the future [38].

3.2.1 Lorenz Model

The Lorenz model exhibits a chaotic dynamics [39]. Using the notation

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the Lorenz vector field reads

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \sigma(y-x) \\ x(r-z) - y \\ -bz + xy \end{pmatrix}$$

where we choose the classical parameter values $\sigma = 10, r = 28, b = 8/3$. For these parameters the Lorenz system fulfills our assumption 1 on dissipativity.

Proposition 1 *The trajectories of the Lorenz eventually enter the absorbing domain*

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^3 : rx^2 + \sigma y^2 + \sigma(z-2r)^2 < \frac{b^2 r^2}{b-1} \right\}$$

Proof: Consider the function

$$V(\mathbf{x}) = (\mathbf{x} - \mathbf{a})^\dagger \mathbf{Q}(\mathbf{x} - \mathbf{a})$$

where $\mathbf{a} = (0, 0, 2r)$ and $\mathbf{Q} = \text{diag}(r, \sigma, \sigma)$, note that the matrix is positive-definite. The goal is to find a bounded region Ω – defined by means of a level set of V – such that $V' < 0$ in the exterior of Ω and then apply Theorem 3. To this end, we compute the derivative,

$$\begin{aligned} V'(\mathbf{x}) &= 2(\mathbf{x} - \mathbf{a})^\dagger \mathbf{Q}\mathbf{f}(\mathbf{x}) \\ &= 2\sigma rx(y-x) + \sigma yx(r-z) - \sigma y^2 - b\sigma z(z-2r) + \sigma(z-2r)xy \\ &= -2\sigma rx^2 - \sigma y^2 + \sigma yx(r-z) - b\sigma z(z-2r) - \sigma(r-z)xy \\ &= -2\sigma [rx^2 + y^2 + b(z-r)^2 - br^2]. \end{aligned}$$

Consider the ellipsoid E defined by $rx^2 + y^2 + b(z-r)^2 < br^2$, hence, in the exterior of E we have $V' < 0$. Now we take c to be the largest value of V in E , and we define $\Omega = \{\mathbf{x} \in \mathbb{R}^3 : V(\mathbf{x}) < c\}$. The solutions will eventually enter Ω and remain inside since $V' < 0$ in the exterior of Ω , and once the trajectory enters in Ω it never

leaves the set. It remains to obtain the parameter c . This can be done by means of a Lagrange multiplier. After a computation – see Appendix C of Ref. [40] – we obtain $c = b^2 r^2 / (b - 1)$ for $b \geq 2$, and $\sigma \geq 1$. \square

Inside the absorbing set Ω the trajectory accumulates on the chaotic attractor. We have numerically integrated the Lorenz equations using a fourth order Runge-Kutta, the initial conditions are $x(0) = -10$, $y(0) = 10$, $z(0) = 25$. We observe that the trajectory accumulates on the so called Butterfly chaotic attractor [39], see Fig. 3.2.

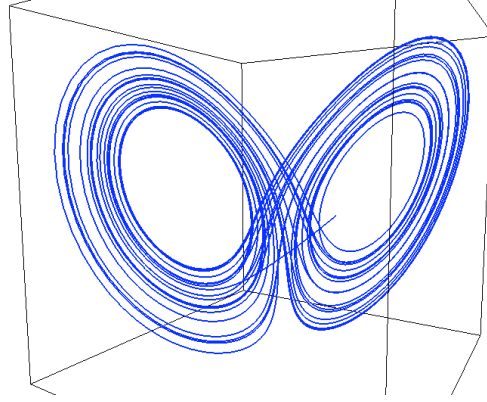


Fig. 3.2 The trajectories of the Lorenz system eventually enters an absorbing domain and accumulates on a chaotic attractor. This projection of attractor resembles a butterfly – the common name of the Lorenz attractor.

Close to the attractor nearby trajectories diverge. To see this phenomenon in a simulation let us consider a distinct initial condition $\tilde{\mathbf{x}}(0) = (\tilde{x}(0), \tilde{y}(0), \tilde{z}(0))^*$. We consider $\tilde{x}(0) = -10.01$, $\tilde{y}(0) = 10$, $\tilde{z}(0) = 25$. Note that the initial difference $\|\mathbf{x}(0) - \tilde{\mathbf{x}}(0)\|_2 = 0.01$ becomes as large as the attractor size in a matter of 6 cycles, see Fig. 3.3.

3.3 Diffusively Coupled Oscillators

We introduce now the network model. On the top of each node of the network we introduce a copy of the system Eq. (3.1). Then the influence that the neighbor j exerts on the dynamics of the node i will be proportional to the difference of their state vector $\mathbf{x}_j(t) - \mathbf{x}_i(t)$. This type of coupling is called diffusive – it tries to equate to state of the nodes.

We label the nodes according to their degrees $k_n \geq \dots \geq k_2 \geq k_1$, where k_1 and k_n denote the minimal and maximal degree, respectively. The dynamics of a network of n identically diffusively coupled elements is described by

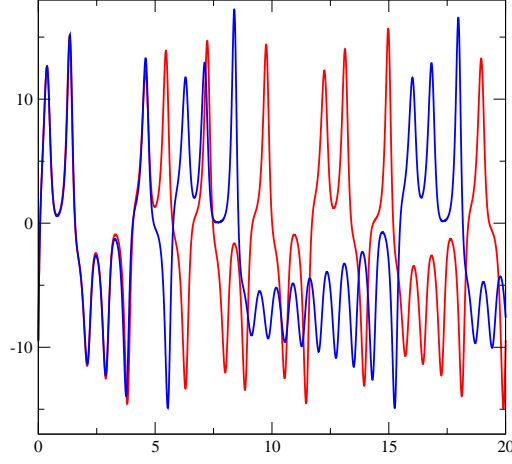


Fig. 3.3 Two distinct simulations of the time series $x(t)$ and $\tilde{x}(t)$ of the Lorentz systems. The difference between the trajectories is of 0.01, however this small difference grows with time until a point where the difference is as large as the attractor itself.

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) + \alpha \sum_{j=1}^n A_{ij}(\mathbf{x}_j - \mathbf{x}_i), \quad (3.2)$$

where α the overall coupling strength. In Eq. (3.2) the coupling is given in terms of the adjacency matrix. We can also represent the coupled equations in terms of the network laplacian. Consider the coupling term

$$\begin{aligned} \sum_{j=1}^n A_{ij}(\mathbf{x}_j - \mathbf{x}_i) &= \sum_{j=1}^n A_{ij}\mathbf{x}_j - k_i\mathbf{x}_i \\ &= \sum_{j=1}^n (A_{ij} - \delta_{ij}k_i)\mathbf{x}_j \end{aligned}$$

where δ_{ij} is the Kronecker delta, and recalling that $L_{ij} = \delta_{ij}k_i - A_{ij}$ we obtain Hence, the equations read

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \alpha \sum_{j=1}^n L_{ij}\mathbf{x}_j. \quad (3.3)$$

The dynamics of such diffusive model can be intricate. Indeed, even if the isolated dynamics possesses a globally stable fixed point the diffusive coupling can

lead to the instability of the fixed points and the systems can exhibit an oscillatory behavior. Please, see [41] for a discussion and further material. We will not focus on such scenario of instability, but rather on how the diffusive coupling can lead to synchronization.

Note that due to the diffusively nature of the coupling, if all oscillators start with the same initial condition the coupling term vanishes identically. This ensures that the globally synchronized state

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_n(t) = \mathbf{s}(t),$$

is an invariant state for all coupling strength α . The question is then the stability of synchronized solutions, which takes place due to coupling. Note that, if $\alpha = 0$ the oscillators are decoupled, and Eq. (3.3) describes n copies of the same oscillator with distinct initial conditions. Since, the chaotic behavior leads to a divergence of nearby trajectories, without coupling, any small perturbation on the globally synchronized motion will grow exponentially fast, and lead to distinct behavior between the node dynamics.

The correct way to see the invariant of globally synchronized motion is as follows. First consider

$$\mathbf{X} = \text{col}(\mathbf{x}_1, \cdots, \mathbf{x}_n),$$

where col denotes the vectorization formed by stacking the columns vectors \mathbf{x}_i into a single column vector. Similarly

$$\mathbf{F}(\mathbf{X}) = \text{col}(\mathbf{f}(\mathbf{x}_1), \cdots, \mathbf{f}(\mathbf{x}_n)),$$

then Eq. (3.3) can be arranged into a compact form

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) - \alpha(\mathbf{L} \otimes \mathbf{I}_m)\mathbf{X} \quad (3.4)$$

where \otimes is the Kronecker product, see Appendix A. Let $\Phi(\cdot, t)$ be the flow of Eq. (3.4), the solution of the equation with initial condition \mathbf{X}_0 is given by $\mathbf{X}(t) = \Phi(\mathbf{X}_0, t)$. Consider the synchronization manifold

$$\mathcal{M} = \{\mathbf{x}_i \in \mathbb{R}^m : \mathbf{x}_i(t) = \mathbf{s}(t) \text{ for } 1 \leq i \leq n\},$$

then we have the following result

Proposition 2 \mathcal{M} is an invariant manifold under the flow $\Phi(\cdot, t)$

Proof: Recall that $\mathbf{I} \in \mathbb{R}^n$ is such that every component is equal to 1. Let $\mathbf{X}(t) = \mathbf{I} \otimes \mathbf{s}(t)$, note that

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{I} \otimes \frac{d\mathbf{s}(t)}{dt}.$$

We claim that $\mathbf{X}(t)$ is a solution of the equations of motion.

$$\begin{aligned}
\frac{dX(t)}{dt} &= F(X(t)) - \alpha(L \otimes I_m)X(t) \\
&= F(I \otimes s(t)) - \alpha(L \otimes I_m)I \otimes s(t) \\
&= I \otimes f(s(t))
\end{aligned}$$

where in the last passage we used Theorem () together with $L I = 0$ and $F(I \otimes s(t)) = I \otimes f(s(t))$. By the Picard-Lindelöf Theorem 16 we have that that $X(t) = \Phi(I \otimes s(0), t) \in \mathcal{M}$ for all t . \square

If the oscillators have the same initial condition, their evolution will be exactly the same forward in time no matter the value of the coupling strength.

In the above result we have looked at the network not as coupled equation but as a single system in the full state space \mathbb{R}^{mn} . We prefer to keep the picture of coupled oscillators. These pictures are equivalent and we interchange them whenever it suits our purposes. The important questions are

Boundedness of the solutions $x_i(t)$.

Stability of the globally synchronized state (synchronization manifold).

We wish to address the local stability of the globally synchronized state. That is, if all trajectories start close together $\|x_i(0) - x_j(0)\| \leq \varepsilon$, for any i and j and some small ε , would they converge to \mathcal{M} , in other words, would

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0$$

or would the trajectories split apart? The goal of the remaining exposition is to provide positive answers to these questions. To this end, we review some fundamental results needed to address such points.

Chapter 4

Linear Differential Equations

The more you know, the less sure you are.
– Voltaire

The question concerning the local stability of a given trajectory $s(t)$ leads to the stability analysis of the trivial solution of a nonautonomous linear differential equation. The analysis of the dynamics in a neighborhood of the solutions is performed by using the variational equation. The trajectory $s(t)$ is stable when the stability of the trivial solution of the variational equation is preserved under small perturbations.

4.1 First Variational Equation

Let $y(0)$ be close to $s(0)$. Each of these distinct points has its behavior determined by the equation of motion Eq. (3.1). We can follow the dynamics of the difference

$$z(t) = y(t) - s(t)$$

which leads to the variational equations governing its evolution

$$\begin{aligned}\frac{dz(t)}{dt} &= f(y(t)) - f(s(t)) \\ &= f(s(t) + z(t)) - f(s(t)),\end{aligned}$$

now since $\|z\|$ is sufficiently small we may expand the function f in Taylor series

$$f(s(t) + z(t)) = f(s(t)) + Df(s(t))z(t) + R(z(t))$$

where $Df(s(t))$ along the trajectory $s(t)$, and by the Lagrange theorem [42]

$$\|R(z(t))\| = O(\|z(t)\|^2).$$

Truncating the evolution equation of z , up to the first order, we obtain the first variational equation

$$\frac{dz}{dt} = Df(s(t))z.$$

Note that the above equation is non-autonomous and linear. Moreover, since $s(t)$ lies in a compact set and f is continuously differentiable, by Weierstrass Theorem [42], $Df(s(t))$ is a bounded matrix function. If $\|z(t)\| \rightarrow 0$ the two distinct solutions converge to each other and have an identical evolution.

The first variational equation plays a fundamental role to tackling the local stability problem. Suppose that somehow we have succeeded to demonstrate that the trivial solution of the first variational equation is stable. Note that this does not completely solve our problem, because the Taylor remainder acts as a perturbation of the trivial solution. Hence, to guarantee that the problem can be solved in terms of the variational equation we must also obtain conditions on the persistence of the stability of trivial solution under small perturbation. There is a beautiful and simple, yet general, criterion based on uniform contractions. We follow closely the exposition in Ref. [43, 44].

4.2 Stability of Trivial Solutions

Consider the linear differential equation

$$\frac{dx}{dt} = U(t)x \tag{4.1}$$

where $U(t)$ is a continuous bounded linear operator on \mathbb{R}^q for each $t \geq 0$.

The point $x \equiv 0$ is an equilibrium point of the equation Eq. (4.1). Loosely speaking, we say an equilibrium point is locally stable if the initial conditions are in a neighborhood of zero solution remain close to it for all time. The zero solution is said to be locally asymptotically stable if it is locally stable and, furthermore, all solutions starting near 0 tend towards it as $t \rightarrow \infty$.

The time dependence in Eq. (4.1) introduces of additional subtleties [45]. Therefore, we want to state some precise definitions of stability

Definition 1 (Stability in the sense of Lyapunov) *The equilibrium point $x^* = 0$ is stable in the sense of Lyapunov at $t = t_0$ if for any $\varepsilon > 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0$$

Lyapunov stability is a very mild requirement on equilibrium points. In particular, it does not require that trajectories starting close to the origin tend to the origin asymptotically. Also, stability is defined at a time instant t_0 . Uniform stability is a concept which guarantees that the equilibrium point is not losing stability. We insist that for a uniformly stable equilibrium point x^* , δ in the Definition 4.1 not be a function of t_0 , so that equation may hold for all t_0 . Asymptotic stability is made precise in the following definition:

Definition 2 (Asymptotic stability) An equilibrium point $\mathbf{x}^* = 0$ is asymptotically stable at $t = t_0$ if

1. $\mathbf{x}^* = 0$ is stable, and
2. $\mathbf{x}^* = 0$ is locally attractive; i.e., there exists $\delta(t_0)$ such that

$$\|\mathbf{x}(t_0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$$

Definition 3 (Uniform asymptotic stability) An equilibrium point $\mathbf{x}^* = 0$ is uniform asymptotic stability if

1. $\mathbf{x}^* = 0$ is asymptotically stable, and
2. there exists δ_0 independent of t_0 for which equation holds. Further, it is required that the convergence is uniform. That is, for each $\varepsilon > 0$ a corresponding $T = T(\varepsilon) > 0$ such that if $\|\mathbf{x}(s)\| \leq \delta_0$ for some $s \geq 0$ then $\|\mathbf{x}(t)\| < \varepsilon$ for all $t \geq s + T$.

We shall focus on the concept of uniform asymptotic stability. To this end, we wish to express the solutions of the linear equation in a closed form. The theory of differential equations guarantees that the unique solution of the above equation can be written in the form

$$\mathbf{x}(t) = \mathbf{T}(t, s)\mathbf{x}(s)$$

where $\mathbf{T}(t, s)$ is the associated evolution operator [44]. The evolution operator satisfies the following properties

$$\mathbf{T}(t, s)\mathbf{T}(s, u) = \mathbf{T}(t, u)$$

$$\mathbf{T}(t, s)\mathbf{T}(s, t) = \mathbf{I}_m.$$

The following concept plays a major role in these lectures

Definition 4 Let $\mathbf{T}(t, s)$ be the evolution operator associated with Eq. (4.1). $\mathbf{T}(t, s)$ is said to be a uniform contraction if

$$\|\mathbf{T}(t, s)\| \leq Ke^{-\eta(t-s)}.$$

where K and η are positive constants.

Some examples of evolution operators and uniform contractions are

Example 1 If \mathbf{U} is a constant matrix, then Eq. (4.1) is autonomous, and the fundamental matrix reads

$$\mathbf{T}(t, s) = e^{(t-s)\mathbf{U}},$$

$\mathbf{T}(t, s)$ has a uniform contraction if, and only if all its eigenvalues have negative real part.

Example 2 Consider the scalar differential equation

$$x' = \{\sin \log(t+1) + \cos \log(t+1) - b\}x,$$

the evolution operator reads

$$T(t, s) = \exp\{-b(t-s) + (t+1)\sin\log(t+1) - (s+1)\sin\log(s+1)\}$$

Then following holds for the equilibrium point $x = 0$

- i) If $b < 1$, the equilibrium is unstable.
- ii) If $b = 1$, the equilibrium is stable but not uniformly stable.
- iii) If $1 < b < \sqrt{2}$, the equilibrium is asymptotically stable but not uniformly stable or uniformly asymptotically stable.
- iv) If $b = \sqrt{2}$, the equilibrium is asymptotically stable. Though it is uniformly stable, it is not uniformly asymptotically stable.
- v) If $b > \sqrt{2}$, the equilibrium is uniformly asymptotically stable.

We will show that the trivial solution of Eq. 4.1 is uniformly asymptotically stable if, and only if, the evolution operator is a uniform contraction, that is, the solutions converge exponentially fast to zero.

Theorem 4 *The trivial solution of Eq. (4.1) is uniformly asymptotic stable if, and only if the evolution operator is a uniform contraction.*

Proof: First suppose the evolution operator is a uniform contraction then

$$\begin{aligned} \|\mathbf{x}(t)\| &= \|T(t, s)\mathbf{x}(s)\| \\ &\leq \|T(t, s)\| \|\mathbf{x}(s)\| \\ &\leq Ke^{-\alpha(t-s)} \|\mathbf{x}(s)\|. \end{aligned}$$

Now let $\varepsilon > 0$ be given, clearly if $t > T$, where $T = T(\varepsilon)$ is large enough then the $\|\mathbf{x}(t)\| \leq \varepsilon$. Let $\|\mathbf{x}(s)\| \leq \delta$, we obtain $\|\mathbf{x}(t)\| \leq Ke^{-\alpha(t-s)} \delta < \varepsilon$, which implies that

$$T = T(\varepsilon) = \frac{1}{\alpha} \ln \frac{\delta K}{\varepsilon},$$

completing the first part.

To prove the converse, we assume that the trivial solution is uniformly asymptotically stable. Then there is δ such that for any ε and $T = T(\varepsilon)$ such that for any $\|\mathbf{x}(s)\| \leq \delta$ we have

$$\|\mathbf{x}(t)\| \leq \varepsilon,$$

for any $t \geq s + T$. Now take $\varepsilon = \delta/k$, and consider the sequence $t_n = s + nT$.

Note that

$$\|T(t, s)\mathbf{x}(s)\| \leq \frac{\delta}{k},$$

for any $\|\mathbf{x}(s)\|/\delta \leq 1$, we have the following bound for the norm

$$\|T(t, s)\| = \sup_{\|\mathbf{u}\| \leq 1} \|T(t, s)\mathbf{u}\| \leq \frac{1}{k}.$$

Remember that $\mathbf{T}(t, u)\mathbf{T}(u, s) = \mathbf{T}(t, s)$. Hence,

$$\begin{aligned}\|\mathbf{T}(t_2, s)\| &= \|\mathbf{T}(s + 2T, s + T)\mathbf{T}(s + T, s)\| \\ &\leq \|\mathbf{T}(s + 2T, s + T)\| \|\mathbf{T}(s + T, s)\| \\ &\leq \frac{1}{k^2}.\end{aligned}$$

Likewise, by induction

$$\|\mathbf{T}(t_n, s)\| \leq \frac{1}{k^n},$$

take $\alpha = \ln k/T$, therefore,

$$\|\mathbf{T}(t_n, s)\| \leq e^{-\alpha(t_n - s)}.$$

Consider the general case $t = s + u + nT$, where $0 \leq u < T$, then the same bound holds

$$\begin{aligned}\|\mathbf{T}(t, s)\| &\leq e^{-nT\alpha} \\ &\leq Ke^{-(t-s)\alpha},\end{aligned}$$

where $K \leq e^{\alpha T}$, and we conclude the desired result. \square

4.3 Uniform Contractions and Their Persistence

The uniform contractions have a rather important roughness property, they are not destroyed under perturbations of the linear equations.

Proposition 3 Suppose $\mathbf{U}(t)$ is a continuous matrix function on \mathbb{R}_+ and consider Eq. (4.1). Assume the fundamental matrix $\mathbf{T}(t, s)$ has a uniform contraction. Consider a continuous matrix function $\mathbf{V}(t)$ satisfying

$$\sup_{t \geq 0} \|\mathbf{V}(t)\| = \delta \leq \frac{\eta}{K}$$

then the evolution operator $\hat{\mathbf{T}}(t, s)$ of the perturbed equation

$$\frac{d\mathbf{y}}{dt} = [\mathbf{U}(t) + \mathbf{V}(t)]\mathbf{y},$$

also has a uniform contraction satisfying

$$\|\hat{\mathbf{T}}(t, s)\| \leq Ke^{-\gamma(t-s)},$$

where $\gamma = \eta - \delta K$.

Proof: Let us start by noting that the evolution operator $T(t, s)$ also satisfies the differential equation of the unperturbed problem

$$\frac{d}{dt}T(t, s) = U(t)T(t, s),$$

The evolution operator \hat{T} can be obtain by the variation of parameter, see Ap. B Theorem 18. So,

$$\hat{T}(t, s) = T(t, s) + \int_s^t T(t, u) V(u) \hat{T}(u, s) du,$$

using the induce norm, for $t \geq s$,

$$\|\hat{T}(t, s)\| \leq Ke^{-\eta(t-s)} + \delta K \int_s^t e^{-\eta(t-u)} \|\hat{T}(u, s)\| du.$$

Let us introduce the scalar function $w(u) = e^{-\eta(t-u)} \|\hat{T}(t, s)\|$, then

$$w(t) \leq Kw(s) + K\delta \int_s^t w(u) du,$$

for all $t \geq s$. Now we can use the Gronwall's inequality to estimate $w(t)$, see Ap. B Theorem 2, this implies

$$w(t) \leq Kw(s)e^{\delta K(t-s)},$$

consequently

$$\|\hat{T}(t, s)\| \leq Ke^{(\eta - K\delta)(t-s)}.$$

□

The roughness property of uniform contraction does the job and guarantees that the stability of the trivial solution is maintained. The question now turns to how to obtain a criterion for uniform contractions. There are various criteria, and the following suits our purposes

Lemma 1 (Principle of Linearization) *Assume the the fundamental matrix $T(t, s)$ of Eq. (4.1) has a uniform contraction. Consider the perturbed equation*

$$\frac{d\mathbf{y}}{dt} = U(t)\mathbf{y} + \mathbf{R}(\mathbf{y}),$$

and assume that

$$\|\mathbf{R}(\mathbf{y})\| \leq M\|\mathbf{y}\|^{1+c},$$

for some $c > 0$. Then the origin is exponentially asymptotically stable.

Proof: Note that we can write $\|\mathbf{R}(\mathbf{y})\| \leq K\|\mathbf{y}\|^1$, where $K = M\|\mathbf{y}\|$. Now given a neighborhood of the trivial solution $\|\mathbf{y}\| \leq \delta$ is possible to control $K \leq \varepsilon$. Applying the previous Proposition 3 we conclude the result. □

This result can be used to prove that if the origin of a nonlinear system is uniformly asymptotically stable then the linearized system about the origin describes the behavior of the nonlinear system.

4.4 Criterion for Uniform Contraction

The question now concerns the criteria to obtain a uniform contraction. There are many results in this direction, we suggest Ref. [43]. We present a criterion that best suits our purpose. The criterion provides a condition only in terms of the equation, and requires no knowledge of the solutions.

Theorem 5 *Let $\mathbf{U}(t) = [U_{ij}(t)]$ be a bounded, continuous matrix function on \mathbb{R}^m on the half-line and suppose there exists a constant $\eta > 0$ such that*

$$U_{ii}(t) + \sum_{\substack{j=1, \\ j \neq i}}^m |U_{ij}(t)| \leq -\eta < 0, \quad (4.2)$$

for all $t \geq 0$ and $i = 1, \dots, m$. Then the evolution operator is a uniform contraction.

Proof: We use the norm $\|\cdot\|_\infty$ and its induced norm, see Ex 4 in Ap. A. Let $\mathbf{x}(t)$ be a solution. For a fixed time $u > 0$ and let $\|\mathbf{x}(u)\|_\infty^2 = x_i(u)^2$. Note $\mathbf{x}(t)$ is a differentiable function and the norm a continuous function $x_i(t)^2$ will also be the norm in an open interval $I = (u-a, u+a)$ for some $a > 0$. Therefore,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{x}(t)\|_\infty^2 &= \frac{1}{2} \frac{d}{dt} [x_i(t)]^2 \\ &= x_i(t) \left(\sum_{j=1}^m U_{ij} x_j \right) \\ &= U_{ii}(t) x_i^2(t) + \sum_{\substack{j=1, \\ j \neq i}}^m U_{ij} x_i(t) x_j(t) \\ &\leq U_{ii}(t) x_i^2(t) + \sum_{\substack{j=1, \\ j \neq i}}^m |U_{ij}(t)| x_i^2(t), \end{aligned}$$

and consequently,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{x}(t)\|_\infty^2 \leq \left(U_{ii}(t) + \sum_{\substack{j=1, \\ j \neq i}}^m |U_{ij}(t)| \right) \|\mathbf{x}(t)\|_\infty^2.$$

Using the condition

$$U_{ii}(t) + \sum_{\substack{j=1, \\ j \neq i}}^m |U_{ij}(t)| \leq -\eta < 0, \quad (4.3)$$

replacing in the inequality

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{x}(t)\|_\infty^2 \leq -\eta \|\mathbf{x}(t)\|_\infty^2,$$

an integration yields

$$\|\mathbf{x}(t)\|_\infty^2 \leq \|\mathbf{x}(s)\|_\infty^2 - 2\eta \int_s^t \|\mathbf{x}(\tau)\|_\infty^2 d\tau,$$

for all $t, s \in I$ and $t > s$. Applying the Gronwall inequality we have which implies

$$\|\mathbf{x}(t)\|_\infty \leq e^{-\eta(t-s)} \|\mathbf{x}(s)\|_\infty. \quad (4.4)$$

Next note that the argument does not depend on the particular component i , because we assume that Eq. (4.3) is satisfied for any $1 \leq i \leq m$. So the norm will satisfy the bound in Eq. 4.4 for any compact set of \mathbb{R}_+ . Noting that all norms are equivalent in finite dimensional spaces the result follows \square .

Chapter 5

Stability of Synchronized Solutions

*Things which have nothing in common cannot be understood,
the one by means of the other; the conception of one does not
involve the conception of the other*

— Spinoza

We come back to the two fundamental questions concerning the boundedness of the solutions and the stability of the globally synchronized in networks of diffusively coupled oscillators.

5.1 Global Existence of the solutions

The remarkable property of the networks of diffusively coupled dissipative oscillators is that the solutions are always bounded, regardless the coupling strength and network structure. The two main ingredients for such boundedness of solutions are:

- Dissipation of the isolated dynamics given in terms of the Lyapunov function V .
- Diffusive coupling given in terms of the laplacian matrix

Under these two conditions we can construct a Lyapunov function for the whole system. The result is then the following

Theorem 6 *Consider the diffusively coupled network model*

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \alpha \sum_{j=1}^n L_{ij} \mathbf{x}_j,$$

and assume that the isolated system has a Lyapunov function satisfying Assumption 1. Then, for any network the solutions of the coupled equations eventually enter an absorbing domain Ω . The absorbing set is independent of the network.

Proof: The idea is to construct a Lyapunov function for the coupled oscillators in terms of the Lyapunov function of the isolated oscillators. Consider the function $W : \mathbb{R}^{mn} \rightarrow \mathbb{R}$ where

$$W(\mathbf{X}) = \frac{1}{2}(\mathbf{X} - \mathbf{A})^* (\mathbf{I}_n \otimes \mathbf{Q})(\mathbf{X} - \mathbf{A})$$

where \mathbf{X} is given by the vectorization of $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ and likewise $\mathbf{A} = (\mathbf{I} \otimes \mathbf{a})^*$, where again $\mathbf{I} = (1, \dots, 1)$. The derivative of the function W along the solutions reads

$$\begin{aligned} \frac{dW(\mathbf{X})}{dt} &= (\mathbf{X} - \mathbf{A})^* (\mathbf{I}_n \otimes \mathbf{Q}) [\mathbf{F}(\mathbf{X}) - \alpha (\mathbf{L} \otimes \mathbf{I}_m) \mathbf{X}] \\ &= (\mathbf{X} - \mathbf{A})^* (\mathbf{I}_n \otimes \mathbf{Q}) \mathbf{F}(\mathbf{X}) - \alpha \mathbf{X}^* (\mathbf{L} \otimes \mathbf{Q}) \mathbf{X} + \alpha \mathbf{A}^* (\mathbf{L} \otimes \mathbf{Q}) \mathbf{X}, \end{aligned}$$

however, using the properties of the Kronecker product, see Theorem 10 and Theorem 11 we have

$$\mathbf{A}^* (\mathbf{L} \otimes \mathbf{Q}) = (\mathbf{I} \otimes \mathbf{a})^* (\mathbf{L} \otimes \mathbf{Q}) \quad (5.1)$$

$$= \mathbf{I}^* \mathbf{L} \otimes \mathbf{a}^* \mathbf{Q} \quad (5.2)$$

but since \mathbf{I} is an eigenvector with eigenvalue 0 we have $\mathbf{I}^* \mathbf{L} = \mathbf{0}^*$, and consequently

$$\mathbf{A}^* (\mathbf{L} \otimes \mathbf{Q}) \mathbf{X} = 0. \quad (5.3)$$

Now \mathbf{L} is positive semi-definite and \mathbf{Q} is positive definite, hence it follows that $\mathbf{L} \otimes \mathbf{Q}$ is positive semi-definite, see Theorem 14, and

$$\mathbf{X}^* (\mathbf{L} \otimes \mathbf{Q}) \mathbf{X} \geq 0.$$

We have the following upper bound

$$\begin{aligned} \frac{dW(\mathbf{X})}{dt} &\leq (\mathbf{X} - \mathbf{A})^* (\mathbf{I}_n \otimes \mathbf{Q}) \mathbf{F}(\mathbf{X}) \\ &= \sum_{i=1}^n (\mathbf{x}_i - \mathbf{a})^* \mathbf{Q} \mathbf{f}(\mathbf{x}_i) \end{aligned} \quad (5.4)$$

$$= \sum_{i=1}^n V'(\mathbf{x}_i) \quad (5.5)$$

but by hypothesis $(\mathbf{x}_i - \mathbf{a})^* \mathbf{Q} \mathbf{f}(\mathbf{x}_i)$ is negative on $D \setminus \Omega$, hence, dW/dt is negative on $D^n \setminus \Omega^n$, since Ω depends only on the isolated dynamics the result follows. \square

This means that the trajectory of each oscillators is bounded

$$\|\mathbf{x}_i(t)\| \leq K$$

where K is a constant and can be chosen to be independent of the node i and of the network parameters such as degree and size.

5.2 Trivial example: Autonomous linear equations

Before we study the stability of the synchronized motion in networks of nonlinear equations, we address the stability problem between two mutually coupled linear equations. The following example is pedagogic and bears all the ideas of the prove of the general case. Consider the scalar equation

$$\frac{dx}{dt} = ax$$

where $a > 0$. The evolution operator reads

$$T(t, s) = e^{a(t-s)},$$

so solutions starting at x_0 are given by $x(t) = e^{at}x_0$. The dynamics is rather simple, for all initial conditions $x_0 \neq 0$ diverge exponentially fast with rate of divergency given by a . Consider two of such equations diffusively coupled

$$\begin{aligned}\frac{dx_1}{dt} &= ax_1 + \alpha(x_2 - x_1) \\ \frac{dx_2}{dt} &= ax_2 + \alpha(x_1 - x_2)\end{aligned}$$

The pain in the neck is that the solutions of the isolated system are not bounded. Since the equation is linear the nontrivial solution are not bounded. On the other hand, because the linearity we don't need the boundedness of solutions to address synchronization. If α is large enough the two systems will synchronize

$$\lim_{t \rightarrow \infty} |x_1(t) - x_2(t)| = 0.$$

Let us introduce

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The adjacency matrix and Laplacian are given

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The coupled equations can be represented as

$$\frac{dX}{dt} = [aI_2 - \alpha L]X$$

According to the example 1 the solution reads

$$X(t) = e^{[aI_2 - \alpha L]t} X_0. \quad (5.6)$$

We can compute the eigenvalues and eigenvectors of the Laplacian \mathbf{L} . An easy computation shows that $\mathbf{I} = (1, 1)^*/\sqrt{2}$ is an eigenvector associated with the eigenvalue 0, and $\mathbf{v}_2 = (1, -1)/\sqrt{2}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 2$. Note that with respect to the Euclidean inner product the set $\{\mathbf{I}, \mathbf{v}_2\}$ is an orthonormal basis of \mathbb{R}^2 .

To solve Eq. (5.6) we note that if for a given matrix \mathbf{B} we have that \mathbf{u} is an eigenvector associated with the eigenvalue λ . Then the matrix $\mathbf{C} = \mathbf{B} - a\mathbf{I}$ has eigenvector \mathbf{u} associated with the eigenvalue $\lambda - a$.

We can write

$$\mathbf{X}_0 = c_1\mathbf{I} + c_2\mathbf{v}_2,$$

recalling that $e^{\mathbf{B}t}\mathbf{u} = e^{\lambda t}\mathbf{u}$, and if \mathbf{B} and \mathbf{C} commute then $e^{\mathbf{B}+\mathbf{C}} = e^{\mathbf{B}}e^{\mathbf{C}}$. Hence, the solution of the vector equation $\mathbf{X}(t)$ reads

$$\mathbf{X}(t) = e^{[a\mathbf{I} - \alpha\mathbf{L}]t} (c_1\mathbf{I} + c_2\mathbf{v}_2) \quad (5.7)$$

$$= c_1e^{at}\mathbf{I} + c_2e^{(a-\alpha\lambda_2)t}\mathbf{v}_2. \quad (5.8)$$

To achieve synchronization the dynamics along the transversal mode \mathbf{v}_2 must be damped out, that is, $\lim_{t \rightarrow \infty} c_2e^{(a-\alpha\lambda_2)t}\mathbf{v}_2 = 0$. This implies that

$$\alpha\lambda_2 > a \Rightarrow \alpha > \frac{a}{\lambda_2}$$

Hence, the coupling strength has to be larger than the rate of divergence of the trajectories over the spectral gap. This is a general principle in diffusively networks.

5.3 Two coupled nonlinear equations

Let us consider now the stability of two oscillators diffusively coupled. At this time we perform the stability analysis without using the Laplacian properties. This allows a simple analysis and provides the condition for synchronization in the same spirit as we shall use later on.

We assume that the nodes are described by Eq. (3.1). In the simplest case of two diffusively coupled in all variables systems the dynamics is described by

$$\begin{aligned} \frac{d\mathbf{x}_1}{dt} &= \mathbf{f}(\mathbf{x}_1) + \alpha(\mathbf{x}_2 - \mathbf{x}_1) \\ \frac{d\mathbf{x}_2}{dt} &= \mathbf{f}(\mathbf{x}_2) + \alpha(\mathbf{x}_1 - \mathbf{x}_2) \end{aligned}$$

where α is the coupling parameter. Again, note that

$$\mathbf{x}_1(t) = \mathbf{x}_2(t)$$

defines the synchronization manifold and is an invariant subspace of the equations of motion for all values of the coupling strength. Note that in the subspace the coupling term vanishes, and the dynamics is the same as if the systems were uncoupled. Hence, we do not control the motion on the synchronization manifold. If the isolated oscillators possess a chaotic dynamics, then the synchronized motion will also be chaotic.

Again, the problem is then to determine the stability of such subspace in terms of the coupling parameter, the coupling strength. It turns out that the subspace is stable if the coupling is strong enough. That is, the two oscillators will synchronize. Note that when they synchronize they will preserve the chaotic behavior.

To determine the stability of the synchronization manifold, we analyze the dynamics of the difference $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2$. Our goal is to obtain conditions such that

$$\lim_{t \rightarrow \infty} \mathbf{z} = \mathbf{0},$$

hence, we aim at obtaining the first variational for \mathbf{z} .

$$\frac{d\mathbf{z}(t)}{dt} = \frac{d\mathbf{x}_1(t)}{dt} - \frac{d\mathbf{x}_2(t)}{dt} \quad (5.9)$$

$$= \mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) - 2\alpha\mathbf{z} \quad (5.10)$$

Now if $\|\mathbf{z}(0)\| \ll 1$, we can obtain the first variational equation governing the perturbations

$$\frac{d\mathbf{z}(t)}{dt} = [\mathbf{Df}(\mathbf{x}_1(t)) - 2\alpha\mathbf{I}]\mathbf{z}. \quad (5.11)$$

The solutions of the variational equation can be written in terms of the evolution operator

$$\mathbf{z}(t) = \mathbf{T}(t, s)\mathbf{z}(s)$$

Applying Theorem 5 we obtain conditions for the evolution operator to possess a uniform contraction. Let us denote the matrix $\mathbf{Df}(\mathbf{x}_1(t)) = [\mathbf{Df}(\mathbf{x}_1(t))_{ij}]_{i,j=1}^n$. Uniform contraction requires

$$\mathbf{Df}(\mathbf{x}_1(t))_{ii} - 2\alpha + \sum_{j=1, j \neq i}^m |\mathbf{Df}(\mathbf{x}_1(t))_{ij}| < 0 \quad (5.12)$$

for all $t \geq 0$, similarly

$$\alpha_c = \sup_{\substack{\mathbf{x} \in \Omega, \\ 1 \leq i \leq m}} \left\{ \sum_{\substack{j=1, \\ j \neq i}}^m |\mathbf{Df}(\mathbf{x}(t))_{ij}| + \mathbf{Df}(\mathbf{x}(t))_{ii} \right\}, \quad (5.13)$$

since Ω is limited and connected in virtue of the Weierstrass Theorem α_c exists. Note that α_c is closely related to the norm of the Jacobian $\|\mathbf{Df}(\mathbf{x})\|_\infty$. Interestingly, α_c can be computed only by accessing the absorbing domain and the Jacobian. Note

that this bound for critical coupling is usually larger than needed to observe synchronization. However, this bound is general and independent of the trajectories, and guarantee a stable and robust synchronized motion.

The trivial solution $z \equiv 0$ might be stable before we guarantee that the evolution operator is a uniform contraction. In this case, however, we don't guarantee that stability of the trivial solutions persists under perturbations. Hence, we cannot guarantee that the nonlinear perturbation coming from the Taylor remainder does not destroy the stability. We avoid tackling this case, since it would bring only further technicalities. Note the above α_c synchronization is stable under small perturbations

Example 3 Consider the Lorenz system presented in Sec. 3.2.1.

Then

$$[Df(x) - \alpha I_3] = \begin{pmatrix} -\sigma - \alpha & \sigma & 0 \\ r - z & -\alpha - 1 & -x \\ y & x & -b - \alpha \end{pmatrix},$$

noting that the trajectories lie within the absorbing domain Ω given in Proposition 1, we have

$$|x| \leq \sqrt{r} \frac{b}{\sqrt{b-1}}, |y| \leq r \frac{b}{\sqrt{\sigma(b-1)}}, \text{ and } |z - r| \leq r \left(\frac{b}{\sqrt{\sigma(b-1)}} + 1 \right),$$

therefore,

$$\alpha_c = r \left(1 + \frac{b}{\sqrt{\sigma(b-1)}} \right) + \sqrt{r} \frac{b}{\sqrt{b-1}} - 1$$

For the standard parameters (see Sec. 3.2.1) we have $\alpha_c \approx 56.2$. For the two coupled Lorenz, this provides the critical parameter for synchronization

$$\alpha \geq \alpha_c/2 \approx 28.1$$

We have simulated the dynamics of Eq. (5.10) using the Lorenz system. For $\alpha = 27 > \alpha_c$ we observe that the complete synchronized state is stable. If the two Lorenz systems start at distinct initial condition as time evolves the difference vanishes exponentially fast, see Fig 5.1

If we depict $x_1 \times x_2$ the dynamics will lie on a diagonal subspace $x = y$. If the initial conditions start away from the diagonal $x = y$ the evolution time series will then converge to it, see Fig 5.2

5.4 Network Global Synchronization

We turn to the stability problem in networks. Basically the same conclusion as before holds: the network is synchronizable for strong enough coupling strengths. In

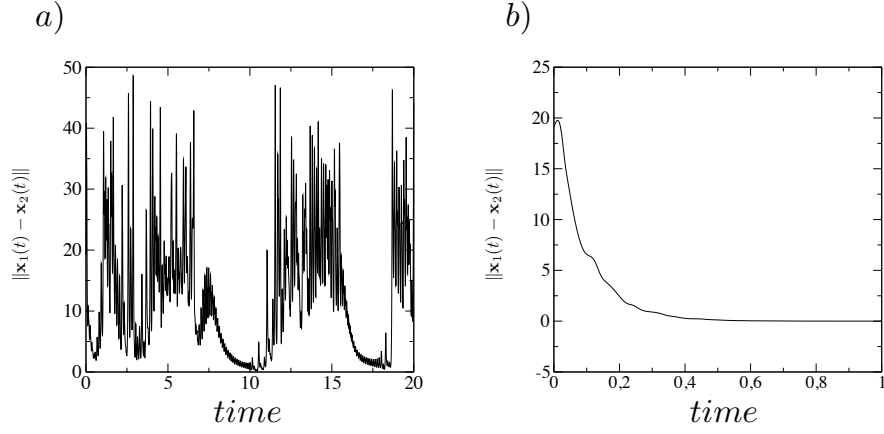


Fig. 5.1 Time evolution of norm $\|x_1(t) - x_2(t)\|$ for distinct initial conditions. *a)* For $\alpha = 0.3$ we observe an asynchronous behavior. *b)* for $\alpha = 27$ above the critical coupling parameter the norm of the difference vanishes exponential fast as a function of times, just as predicted by the uniform contraction.

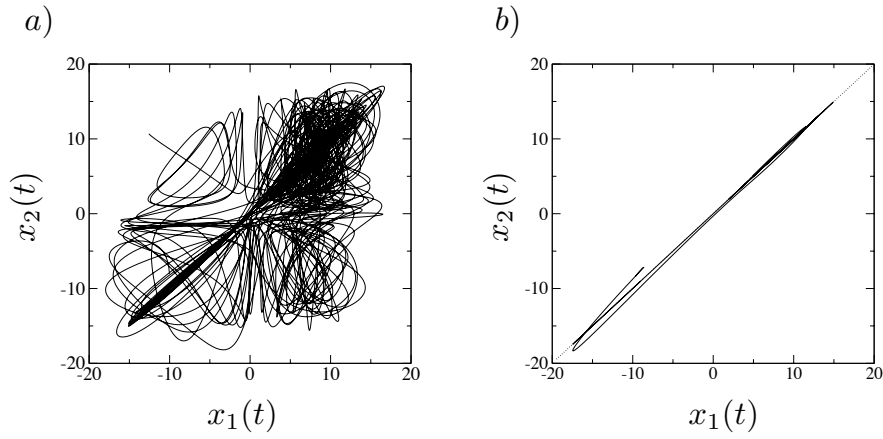


Fig. 5.2 Behavior of the trajectories in the projection $x_1 \times x_2$. *a)* for the coupling parameter $\alpha = 3$, the trajectories are out of sync, and spread around. *b)* for $\alpha = 27$ the trajectories converge to the diagonal line $x_1 = x_2$. Trajectories in a neighborhood of the diagonal converge to it exponentially fast.

such a case we want to determine the critical coupling in relation to the network structure. A positive answer to these question is given by the following

Theorem 7 *Consider the diffusively coupled network model*

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \alpha \sum_{j=1}^n A_{ij}(\mathbf{x}_j - \mathbf{x}_i),$$

on a connected network. Assume that the isolated system has a Lyapunov function satisfying Assumption 1 with an absorbing domain Ω . Moreover, assume that for a given time $s \geq 0$ all trajectories are in a neighborhood of the synchronization manifold lying on the absorbing domain Ω , and consider the α_c given by Eq. (5.13), and λ_2 the smallest nonzero eigenvalue of the Laplacian. Then, for any

$$\alpha > \frac{\alpha_c}{\lambda_2},$$

the global synchronization is uniformly asymptotically stable. Moreover, the transient to the globally synchronized behavior is given the algebraic connectivity, that is, for any i and j

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq M e^{-(\alpha\lambda_2 - \alpha_c)t}$$

The above result relates the threshold coupling for synchronization in contributions coming solely from dynamics α_c , and network structure λ_2 . Therefore, for a fixed node dynamics we can analyze how distinct network facilitates or inhibits global synchronization. To continue our discussion we need the following

Definition 5 *Let $\beta(G)$ be the critical coupling parameter for the network G . We say that the network G is better synchronizable than H if for fixed node dynamics*

$$\beta(G) < \beta(H)$$

Recalling the general bounds presented in Theorem 2 we conclude that the complete network is the most synchronizable network. Furthermore, the following general statement is also true

– *For a fixed network size, network with small diameter are better synchronizable. Hence, the ability of the network to synchronize depends on the overall connectedness of the graph.*

Recall the results presented in table 2.2, and let denote α_c denote the critical coupling parameter, the dependence of α_c in terms of the network size can be seen table 5.4

The difficulty to synchronize a complete network decreases with the network size, whereas to synchronize the cycle increases quadratically with the size.

Now we present the proof of Theorem 7. We must show that the synchronization manifold \mathcal{M} is locally attractive. In other words, whenever the nodes start close

Table 5.1 Leading order dependence of β on the network size for the networks in Fig. 2.2

Network	β
Complete	$\frac{1}{n}$
ring	$\frac{n^2}{2}$
Star	1

together they tend to the same future dynamics, that is, $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \rightarrow 0$, for any i and j . For pedagogical purposes we split the proof into four main steps.

Step 1: Expansion into the Laplacian Eigenmodes. Consider the equations of motion in the block form

$$\frac{dX}{dt} = F(X) - \alpha(L \otimes I_m)X$$

Note that since L is symmetric, by Theorem 9 there exists an orthogonal matrix O such that

$$L = OMO^*,$$

where $M = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_n)$ is the eigenvalue matrix. Introducing

$$Y = \text{col}(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n)$$

we can write the above equation in terms of Laplacian eigenvectors

$$\begin{aligned} X &= (O \otimes I_m)Y, \\ &= \sum_{i=1}^n \mathbf{v}_i \otimes \mathbf{y}_i \end{aligned}$$

For sake of simplicity we call $\mathbf{y}_1 = s$, and remember that now note that $\mathbf{v}_1 = I$ hence

$$X = I \otimes s + U,$$

where

$$U = \sum_{i=2}^n \mathbf{v}_i \otimes \mathbf{y}_i.$$

In this way we split the contribution in the direction of the global synchronization and U , which accounts for the contribution of the transversal. Note that if U converges to zero then the system completely synchronizes, that is X converges to $I \otimes s$ which clearly implies that

$$\mathbf{x}_1 = \dots = \mathbf{x}_n = s$$

The goal then is to obtain conditions so that \mathbf{U} converges to zero.

Step 2: Variational equations for the Transversal Modes. The equation of motion in terms of the Laplacian modes decomposition reads

$$\begin{aligned}\frac{d\mathbf{X}}{dt} &= \mathbf{F}(\mathbf{X}) - \alpha(\mathbf{L} \otimes \mathbf{I})\mathbf{X} \\ \mathbf{I} \otimes \frac{ds}{dt} + \frac{d\mathbf{U}}{dt} &= \mathbf{F}(\mathbf{I} \otimes s + \mathbf{U}) - \alpha(\mathbf{L} \otimes \mathbf{I})(\mathbf{I} \otimes s + \mathbf{U}),\end{aligned}$$

We assume that \mathbf{U} is small and perform a Taylor expansion about the synchronization manifold.

$$\mathbf{F}(\mathbf{I} \otimes s + \mathbf{U}) = \mathbf{F}(\mathbf{I} \otimes s) + D\mathbf{F}(\mathbf{I} \otimes s)\mathbf{U} + \mathbf{R}(\mathbf{U}),$$

where $\mathbf{R}(\mathbf{U})$ is the Taylor remainder $\|\mathbf{R}(\mathbf{U})\| = O(\|\mathbf{U}\|^2)$. Using the Kronecker product properties 10 and the fact that $\mathbf{L}\mathbf{I} = \mathbf{0}$, together with

$$\mathbf{I} \otimes \frac{ds}{dt} = \mathbf{F}(\mathbf{I} \otimes s) = \mathbf{I} \otimes f(s)$$

and likewise

$$D\mathbf{F}(\mathbf{I} \otimes s)\mathbf{U} = [\mathbf{I}_n \otimes Df(s)]\mathbf{U},$$

and we have

$$\frac{d\mathbf{U}}{dt} = [D\mathbf{F}(\mathbf{I} \otimes s) - \alpha(\mathbf{L} \otimes \mathbf{I})]\mathbf{U} + \mathbf{R}(\mathbf{U}), \quad (5.14)$$

Therefore, the first variational equation for the transversal modes reads

$$\frac{d\mathbf{U}}{dt} = [\mathbf{I}_n \otimes Df(s) - \alpha\mathbf{L} \otimes \mathbf{I}_m]\mathbf{U}.$$

The solution of the above equation has a representation in terms of the evolution operator

$$\mathbf{U}(t) = \mathbf{T}(t, s)\mathbf{U}(s)$$

We want to obtain conditions for the trivial solution of the above to be uniformly asymptotically stable, that is, so that the evolution operator is a uniform contraction.

Step 3: Stabilization of the Transversal Modes. Instead of analyzing the full set of equations, we can do much better by projecting the equation into the transversal modes \mathbf{v}_i . Applying $\mathbf{v}_j^* \otimes \mathbf{I}_m$ on the right in the equation for \mathbf{U} , it yields

$$\begin{aligned}\mathbf{v}_j^* \otimes \mathbf{I}_m \left(\sum_{i=2}^n \mathbf{v}_i \otimes \frac{dy_i}{dt} \right) &= \mathbf{v}_j^* \otimes \mathbf{I}_m \left(\sum_{i=2}^n \mathbf{v}_i \otimes Df(s)\mathbf{y}_i - \alpha\lambda_i \mathbf{v}_i \otimes \mathbf{y}_i \right) \\ \sum_{i=2}^n \mathbf{v}_j^* \mathbf{v}_i \otimes \frac{dy_i}{dt} &= \sum_{i=2}^n \mathbf{v}_j^* \mathbf{v}_i \otimes [Df(s) - \alpha\lambda_i \mathbf{I}_m]\mathbf{y}_i\end{aligned}$$

But since \mathbf{v}_i form an orthonormal basis we have

$$\mathbf{v}_j^* \mathbf{v}_i = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Hence, we obtain the equation for the coefficients

$$\frac{d\mathbf{y}_i}{dt} = [D\mathbf{f}(s) - \alpha\lambda_i \mathbf{I}_m] \mathbf{y}_i$$

All blocks have the same form which are different only by λ_i , the i th eigenvalue of L . We can write all the blocks in a parametric form

$$\frac{d\mathbf{u}}{dt} = \mathbf{K}(t)\mathbf{u}, \quad (5.15)$$

where

$$\mathbf{K}(t) = D\mathbf{f}(s(t)) - \kappa \mathbf{I}_m$$

with $\kappa \in \mathbb{R}$. Hence if $\kappa = \alpha\lambda_i$ we have the equation for the i th block. This is just the same type of equation we encounter before in the example of the two coupled oscillators, see Eq. (5.11).

Now obtain conditions for the evolution operator of Eq. (5.15) to possess a uniform contraction. This is done applying the same arguments discussed in Eqs. 5.12 and 5.13. Therefore, the i th block has a uniform contraction if $\alpha\lambda_i > \alpha_c$. Now since the spectrum of the Laplacian is ordered, the condition for all blocks to be uniformly asymptotically stable is

$$\alpha > \frac{\alpha_c}{\lambda_2}$$

which yields a critical coupling value in terms of α_c and λ_2 .

Taking α larger than the critical value we have that all blocks have uniform contractions. Let $\mathbf{T}_i(t, s)$ be the evolution operator of the i th block. Then

$$\begin{aligned} \|\mathbf{y}_i(t)\| &\leq \|\mathbf{T}_i(t, s) \mathbf{y}_i(s)\| \\ &\leq \|\mathbf{T}_i(t, s)\| \|\mathbf{y}_i(s)\|, \end{aligned} \quad (5.16)$$

by applying Theorem 5 we obtain

$$\|\mathbf{y}_i(t)\| \leq K_i e^{-\eta_i(t-s)} \|\mathbf{y}_i(s)\|,$$

where $\eta_i = \alpha\lambda_i - \alpha_c$.

Step 4: Norm Estimates. Using the bounds for the blocks it is easy to obtain a bound for the norm of the evolution operator. Indeed, note that

$$\begin{aligned}\|\mathbf{U}\|_2 &= \left\| \sum_{i=2}^n \mathbf{v}_i \otimes \mathbf{y}_i \right\|_2 \\ &\leq \sum_{i=2}^n \|\mathbf{v}_i\| \|\mathbf{y}_i\|_2\end{aligned}$$

where we have used Theorem 15 (see Ap. A), therefore,

$$\|\mathbf{U}\|_2 \leq \sum_{i=2}^n \|\mathbf{v}_i\| K_i e^{-\eta_i(t-s)} \|\mathbf{y}_i(s)\|$$

Now using that $e^{-\eta_i(t-s)} \leq e^{-(\alpha\lambda_2 - \alpha_c)}$, and applying Theorem 4 we obtain

$$\|\mathbf{T}(t, s)\|_2 \leq M e^{-\eta(t-s)}$$

with $\eta = \alpha\lambda_2 - \alpha_c$ for any $t \geq s$.

By the principle of linearization Lemma 1, we conclude that the Taylor remainder does not affect the stability of the trivial solution, which correspond to the global synchronization.

The claim about the transient is straightforward, indeed note that

$$\|\mathbf{X}(t) - \mathbf{I} \otimes \mathbf{s}(t)\| \leq M e^{-\eta(t-s)} \|\mathbf{U}(s)\|$$

implying that $\|\mathbf{x}_i(t) - \mathbf{s}(t)\| \leq K e^{-\eta(t-s)}$ and

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \|\mathbf{x}_i(t) - \mathbf{s}(t)\| + \|\mathbf{x}_j(t) - \mathbf{s}(t)\|$$

in virtue of the triangular inequality, and we concluding the proof. \square

Chapter 6

Conclusions and Remarks

I have had my results for a long time: but I do not yet know how I am to arrive at them.

— Gauss

We have used stability results from the theory of nonautonomous differential equation to establish conditions for stable global synchronization in networks of diffusively coupled dissipative dynamical systems. Our conditions split the stability condition solely in terms of the isolated dynamics and network eigenvalues.

The condition associated with the dynamics is related to the norm of the Jacobian of the vector field. This reflects that fact that to obtain stable synchronization we need to damp all instabilities appearing in the variational equation. The network condition is given in terms of the graph algebraic connective – the smallest nonzero eigenvalue, which reflects how well connected is the graph.

The dependence of synchronization on only two parameters is due to our hypotheses: *i)* all isolated equations are the same, and *ii)* the coupling between them is mutual and fully diffusive. These assumptions reduce the intrinsic difficulties of the problem and allow rigorous results.

There are other approaches to tackling the stability of the global synchronization. Successful approaches are the construction of a Lyapunov function of the synchronization manifold, see for example Refs. [46, 47, 48], which takes a control view; and the theory of invariant manifolds [49, 50] taking a dynamical system view of synchronization.

Our results have a deeper connection with the previous approach introduced by Pecora and Carrol [26]. They used the theory of Lyapunov exponents, which allow the tackling of general coupling functions. The main drawback is that of obtaining results for the persistence of the global synchronization. This requires establishing results on the continuity of the Lyapunov exponent, which is rather subtle and intricate [51].¹

The approach introduced in these notes follows the steps of the Pecora and Carrol analysis, that is, the local stability analysis of the synchronization manifold, but uses various concepts in stability theory, to establish the persistence results for the global synchronization.

¹ Small perturbations can destabilize a system with negative Lyapunov exponents. To guarantee the persistence under perturbations Lyapunov regularity is required, see Ref. [51].

Appendix A

Linear Algebra

If only I had the theorems! Then I should find the proofs easily enough.

– Bernard Riemann

For this exposition we consider the field F where $F = \mathbb{R}$ the field of real numbers or $F = \mathbb{C}$ the field of complex numbers. We shall closely follow the exposition of Ref. [52]. Consider the set $\text{Mat}(F, n)$ of all square matrices acting on F^n . We start with the following

Definition 6 Let $\mathbf{A} \in \text{Mat}(F, n)$. The set

$$\sigma(\mathbf{A}) := \{\lambda \in \mathbb{C} : \det(\mathbf{A} - \lambda \mathbf{I}) = 0\}$$

is called the spectrum of \mathbf{A} .

The spectrum of \mathbf{A} is constituted of all its eigenvalues. Note by the fundamental theorem of algebra the spectrum has at most n distinct points.

Often, we want to obtain bounds on the localization of eigenvalues on the complex plane. A handy result is provided by the result

Theorem 8 (Gershgorin) Let $\mathbf{A} \in \text{Mat}(\mathbb{C}, n)$, denote $\mathbf{A} = (A_{ij})_{i,j=1}^n$. Let $D(a, \delta)$ denote the ball of radius δ centered at a . For each $1 \leq i \leq n$ let

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^n |A_{ij}|,$$

then every eigenvalue of \mathbf{A} lies within at least one of the balls $D(A_{ii}, R_i)$.

For a proof see Ref. [52] Sec. 10.6.

If $\mathbf{A} \in \text{Mat}(\mathbb{C}, n)$ then we denote its conjugate transpose by \mathbf{A}^* . In case \mathbf{A} is a real valued matrix \mathbf{A}^* denotes the transpose. A matrix is called hermitian if $\mathbf{A} = \mathbf{A}^*$. The following definition is also fundamental

Definition 7 Let $\mathbf{A} \in \text{Mat}(\mathbb{C}, n)$ be a hermitian matrix. It is called positive-semidefinite (or sometimes nonnegative-definite) if

$$\mathbf{x}^* \mathbf{A} \mathbf{x} \geq 0$$

for any $\mathbf{x} \in \mathbb{C}^n$

It follows that a matrix is nonnegative if all its eigenvalues are non negative.

A.1 Matrix space as a normed vector Space

Consider the vector space F^n over the field F . A norm $\|\cdot\|$ on F^n is a function $\|\cdot\| : F^n \rightarrow \mathbb{R}$ satisfying

1. positive definiteness : $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in F^n$ and equality holds iff $\mathbf{x} = \mathbf{0}$
2. Absolute definiteness : $\|\gamma\mathbf{x}\| = |\gamma|\|\mathbf{x}\|$ for all $\mathbf{x} \in F^n$ and $\gamma \in F$
3. Triangle inequality : $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in F^n$

We call the pair $(F^n, \|\cdot\|)$ is called normed vector space. This normed vector space is also a metric space under the metric $d : F^n \times F^n \rightarrow \mathbb{R}$ where $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. We say that d is the metric induced by the norm. In this metric, the norm defines a continuous map from F^n to \mathbb{R} , and the norm is a convex function of its argument. Normed vector spaces are central to the study of linear algebra.

Once we introduce of norm on the vector space F^n , we can also view the $\text{Mat}(F, n)$ as a normed spaces. This can be done by the induced matrix norm which is a natural extension of the notion of a vector norm to matrices. Given a vector norm $\|\cdot\|$ on F^n , we define the corresponding induced norm or operator norm on the space $\text{Mat}(F, n)$ as:

$$\|\mathbf{A}\| = \sup\{\|\mathbf{A}\mathbf{x}\| : \mathbf{x} \in F^n \text{ and } \|\mathbf{x}\| = 1\}$$

It follows from the theory of functions on compact spaces that $\|\mathbf{A}\|$ always exists and it is called induced norm. Indeed, the induced norm defines a norm on $\text{Mat}(F, n)$ satisfying the properties 1-3 and an additional property

$$\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\|\|\mathbf{B}\| \text{ for all } \mathbf{A}, \mathbf{B} \in \text{Mat}(F, n)$$

called sub-multiplicativity. A sub-multiplicative norm on $\text{Mat}(F, n)$ is called matrix norm or operator norm. Note that even though we use the same notation $\|\mathbf{A}\|$ for the norm of \mathbf{A} , this should not be confused with the vector norm.

Example 4 Consider the norm of the maximum $\|\cdot\|_\infty$ on \mathbb{R}^n . Given $\mathbb{R}^n \ni \mathbf{x} = (x_1, \dots, x_n)$, the norm is defined as $\|\mathbf{x}\| = \max_i |x_i|$. Given a matrix $\mathbf{A} = (A_{ij})_{i,j=1}^n$ then

$$\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |A_{ij}|$$

Example 5 Consider the Euclidean norm $\|\cdot\|_2$ on \mathbb{R}^n . Using the notation of the last example, we have

$$\|\mathbf{A}\|_2 = \sqrt{\rho(\mathbf{A}^* \mathbf{A})}$$

where $\rho_{\max}(\mathbf{A}^* \mathbf{A})$ is spectral radius $\mathbf{A}^* \mathbf{A}$.

Recall that two norms $\|\cdot\|'$ and $\|\cdot\|''$ are said to be equivalent if

$$a\|A\|' \leq \|A\|'' \leq b\|A\|'$$

for some positive numbers a, b and for all matrices A . It follows that in finite-dimensional normed vector spaces any two norms are equivalents.

A.2 Representation Theory

We review some fundamental results on matrix representations. A square matrix A is diagonalizable if and only if there exists a basis of F^n consisting of eigenvectors of A . In other words, if the F^n is spanned by the eigenvectors of A . If such a basis can be found, then $P^{-1}AP$ is a diagonal matrix, where P is the eigenvector matrix, each column of P consists of an eigenvector. The diagonal entries of this matrix are the eigenvalues of A . One of the main goals in matrix analysis is to classify the diagonalizable matrices.

In general diagonalization will depend on the properties of F such as whether F is an algebraically closed field. If $F = \mathbb{C}$ then almost every matrix is diagonalizable. In other words, the set $B \subset \text{Mat}(\mathbb{C}, n)$ of non diagonalizable matrices over \mathbb{C} has Lebesgue measure zero. Moreover, the set diagonalizable matrices form a dense subset. Any non diagonalizable matrix, say $Q \in B$ can be approximated by a diagonalizable matrix. Precisely, given $\varepsilon > 0$ there is a sequence $\{A_i\}$ of diagonalizable matrices such that $\|Q - A_i\| < \varepsilon$ for any $i > n_0$.

Let us denote by $*$ the conjugate transpose if $F = \mathbb{C}$ (clearly only transpose if $F = \mathbb{R}$). We first focus on symmetric matrices $A = A^*$ and $F = \mathbb{R}$. It turns out that it is always possible to diagonalize such matrices.

Definition 8 A real square matrix A is orthogonally diagonalizable if there exists an orthogonal matrix P such that $P^*AP = D$ is a diagonal matrix.

Diagonalization of symmetric matrices is guaranteed by the following

Theorem 9 Let A be a real symmetric matrix. Then there exists an orthogonal matrix P such that :

1. $P^*AP = D$ is a diagonal matrix.
2. $D = \text{diag}(\lambda_1, \dots, \lambda_n)$, where λ_i are the eigenvalues of A .
3. The column vectors of P are the eigenvectors of the eigenvalues of A .

For a proof see Ref. [53] Sec. 8.1.

A.3 Kronecker Product

We need several properties of the Kronecker Product to address the stability of the synchronized motion in networks.

Definition 9 Let $\mathbf{A} \in \text{Mat}(F, m \times n)$ and $\mathbf{B} \in \text{Mat}(F, r \times s)$. The Kronecker Product of the matrices A and B and defined as the matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_{11}\mathbf{B} & \cdots & A_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ A_{m1}\mathbf{B} & \cdots & A_{mn}\mathbf{B} \end{pmatrix}$$

The Kronecker product is sometimes called tensor product. Consider now the following examples on the

Example 6 Consider the matrices

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

Then

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a\mathbf{B} & b\mathbf{B} \\ c\mathbf{B} & d\mathbf{B} \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 2a & 3a & 2b & 3b \\ c & 0 & d & 0 \\ 2c & 3c & 2d & 3d \end{pmatrix}$$

Now consider the vectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Then

$$\mathbf{v} \otimes \mathbf{u}(t) = \begin{pmatrix} x(t) \\ y(t) \\ x(t) \\ y(t) \end{pmatrix}$$

We review the basic results we need.

Theorem 10 Let $\mathbf{A} \in \text{Mat}(F, m \times n)$ and $\mathbf{B} \in \text{Mat}(F, r \times s)$ $\mathbf{C} \in \text{Mat}(F, n \times p)$ and $\mathbf{D} \in \text{Mat}(F, s \times t)$. Then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}.$$

The proof can be found in Ref. [52] pg. 408, see Proposition 2. Note that $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) \in \text{Mat}(F, mr \times pt)$. A direct computation leads to the following result

Theorem 11 Let $\mathbf{A} \in \text{Mat}(F, m \times n)$ and $\mathbf{B} \in \text{Mat}(F, r \times s)$, then

$$(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^*$$

By applying Theorem 10 we conclude that following

Theorem 12 *If \mathbf{A} and \mathbf{B} are nonsingular, then*

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}.$$

The following Theorem also plays an important role in the exposition

Theorem 13 *Let $\{\lambda_i\}_{i=1}^r$ be the eigenvalues of $\mathbf{A} \in \text{Mat}(F, n)$ and $\{\mu_i\}_{i=1}^s$ be the eigenvalues of $\mathbf{B} \in \text{Mat}(F, n)$. Then $\mathbf{A} \otimes \mathbf{B}$ has rs eigenvalues*

$$\lambda_1 \mu_1, \dots, \lambda_1 \mu_s, \lambda_2 \mu_1, \dots, \lambda_2 \mu_s, \dots, \lambda_r \mu_s.$$

The proof can be found in Ref. [52] pg. 412. A direct consequence of this result is the following

Theorem 14 *Let \mathbf{A} and \mathbf{B} be positive semi-definite matrices. Then $\mathbf{A} \otimes \mathbf{B}$ is also positive semi-definite.*

Our last result concerns the norms of the Kronecker products

Theorem 15 *Let $\|\cdot\|_p$ be p -norm. Consider $\mathbf{v} \in \mathbb{R}^s$, and $\mathbf{x} \in \mathbb{R}^t$, for $t, s \in \mathbb{N}$. Then*

$$\|\mathbf{v} \otimes \mathbf{x}\|_p = \|\mathbf{v}\|_p \|\mathbf{x}\|_p$$

Appendix B

Ordinary Differential Equations

Let D be an open connected subset of \mathbb{R}^m , $m \geq 1$, and let $\mathbf{G} : D \rightarrow \mathbb{R}^m$ be an autonomous vector field. Consider the problem of finding solutions for the vector differential equation

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}) \quad (\text{B.1})$$

with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. A positive answer to this problem is given by the following

Theorem 16 (Picard-Lindelöf) *Assume that the vector field \mathbf{G} Lipschitz continuous in a neighborhood of \mathbf{x}_0 . Precisely, assume that given $\mathbf{x}_0 \in U \subset D$ there is a constant K_U such that*

$$\|\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})\| \leq K_U \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in U$. Then there exists a unique local solution $\mathbf{x}(t)$ for Eq. (B.1) satisfying $\mathbf{x}(0) = \mathbf{x}_0$.

Note that the solution is local, in the sense that there is small $\kappa > 0$ such that the function $\mathbf{x} : [-\kappa, \kappa] \rightarrow D$ is a solution of the problem with $\mathbf{x}(0) = \mathbf{x}_0$. The question is: How long does such solution exist for? We are interested in the long term behavior of the solutions, so we wish to know under what conditions the solutions exists forward in time. A positive answer is given by extension theorems:

Theorem 17 (Extension) *Let \mathcal{C} be a compact subset of the open set D . Consider Eq. (B.1) and let \mathbf{G} be differentiable. Let $\mathbf{x}_0 \in \mathcal{C}$ and suppose that every solution $\mathbf{x} : [0, \tau] \rightarrow D$ with $\mathbf{x}(0) = \mathbf{x}_0$ lies entirely in \mathcal{C} . Then this solution is defined for all (forward) time $t \geq 0$.*

The proofs of the above theorems can be found in Refs. [54, 55].

B.1 Linear Differential Equations

The evolution operator also determines the behavior of the non homogeneous equation

Theorem 18 *Let $\mathbf{A} : \mathbb{R} \rightarrow \text{Mat}(\mathbb{R}, n)$ and $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$ be continuous function. Consider the perturbed equation*

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t)$$

The solution of the perturbed equation corresponding to the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{y}(t) = \mathbf{T}(t, t_0)\mathbf{x}_0 + \int_{t_0}^t \mathbf{T}(t, s)\mathbf{g}(s)ds$$

where $\mathbf{T}(t, t_0)$ is the evolution operator of the corresponding homogeneous system.

The following inequality is central to obtain various estimates

Lemma 2 (Gronwall) *Consider $U \subset \mathbb{R}_+$ and let $u : U \rightarrow \mathbb{R}$ be continuous and non-negative function. Suppose there exist $C \geq 0$ and $K \geq 0$ such that*

$$u(t) \leq C + \int_0^t Ku(s)ds \tag{B.2}$$

for all $t \in U$, then

$$u(t) \leq Ce^{Kt}.$$

The proof of these results can be found in Ref. [54].

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